

AN OPTIMAL MATRIX INEQUALITY AND ITS APPLICATIONS TO GEOMETRY OF RIEMANNIAN SUBMERSIONS

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ABSTRACT. Motivated by the matrix form of the DDVV conjecture in submanifold geometry which is an optimal inequality involving norms of commutators of several real symmetric matrices and takes an important role in the proof of the well-known Simons inequality for closed minimal submanifolds in spheres, in this paper we first derive a similar optimal inequality of real skew-symmetric matrices, then we apply it to establish a Simons-type inequality for Riemannian submersions, which shows another “evidence” of the duality between submanifold geometry and Riemannian submersions.

1. INTRODUCTION

Let M^n be an immersed submanifold of a real space form $N^{n+m}(c)$ of constant sectional curvature c . Given an orthonormal basis $\{e_1, \dots, e_n\}$ (resp. $\{\xi_1, \dots, \xi_m\}$) of $T_p M$ (resp. $T_p^\perp M$), the normalized scalar curvature ρ and the normal scalar curvature ρ^\perp of M^n at p are defined by

$$\rho = \frac{2}{n(n-1)} \sum_{1 \leq i < j}^n \langle R(e_i, e_j)e_j, e_i \rangle,$$

$$\rho^\perp = \frac{2}{n(n-1)} \left(\sum_{1 \leq i < j}^n \sum_{1 \leq r < s}^m \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{\frac{1}{2}} = \frac{2}{n(n-1)} |R^\perp|,$$

where R and R^\perp are curvature tensors of the tangent and normal bundles of M respectively. Denote by h the second fundamental form and $H = \frac{1}{n} \text{Tr}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$ the mean curvature vector field. The DDVV conjecture raised by [7] says that there is a pointwise inequality among ρ , ρ^\perp and $|H|^2$ as the following:

$$(1.1) \quad \rho + \rho^\perp \leq |H|^2 + c.$$

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Due to the Gauss and Ricci equations, this conjecture can be translated into the following algebraic inequality (cf. [8]):

$$(1.2) \quad \sum_{r,s=1}^m \|[B_r, B_s]\|^2 \leq \left(\sum_{r=1}^m \|B_r\|^2 \right)^2,$$

where $\{B_1, \dots, B_m\}$ are arbitrary real symmetric $(n \times n)$ -matrices, $[\cdot, \cdot]$ is the commutator operator and $\|\cdot\|$ is the standard norm of matrix.

The inequality (1.2) (and thus the DDVV conjecture (1.1)) has been proved independently and differently by [12, 17]. In particular, the equality condition given in [12] shows that the inequality (1.2) is an optimal inequality. As for the classification problem of submanifolds attaining the equality of (1.1) everywhere, we refer to [6] for a big advance. In this paper, by a similar method as in [12], we obtain the following optimal inequality of real skew-symmetric matrices in the form of the inequality (1.2), which has been previously reviewed in the survey paper [13].

Throughout this paper, a $K := O(n) \times O(m)$ action on (B_1, \dots, B_m) means that

$$(P, R) \cdot (B_1, \dots, B_m) := (PB_1P^t, \dots, PB_mP^t) \cdot R, \quad \text{for } (P, R) \in K.$$

Theorem 1.1. *Let B_1, \dots, B_m be $(n \times n)$ real skew-symmetric matrices.*

(i) *If $n = 3$, then we have*

$$\sum_{r,s=1}^m \|[B_r, B_s]\|^2 \leq \frac{1}{3} \left(\sum_{r=1}^m \|B_r\|^2 \right)^2,$$

where the equality holds if and only if under some K action all B_r 's are zero except 3 matrices which can be written as

$$C_1 := \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 := \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix}, \quad C_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & -\lambda & 0 \end{pmatrix}.$$

(ii) *If $n \geq 4$, then we have*

$$\sum_{r,s=1}^m \|[B_r, B_s]\|^2 \leq \frac{2}{3} \left(\sum_{r=1}^m \|B_r\|^2 \right)^2,$$

where the equality holds if and only if under some K action all B_r 's are zero except 3 matrices which can be written as $\text{diag}(D_1, 0)$, $\text{diag}(D_2, 0)$, $\text{diag}(D_3, 0)$, where $0 \in M(n-4)$ is the zero matrix of order $n-4$ and

$$D_1 := \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{pmatrix}, \quad D_2 := \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \\ -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix}, \quad D_3 := \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix}.$$

In sight of the geometric origin of the inequality (1.2), *i.e.*, the DDVV inequality (1.1) in submanifold geometry, we get interested in applications to geometry of this “dual” matrix inequality. Our attention will be focused on the geometry of Riemannian submersions which in some sense is also a “dual” theory of submanifold geometry. It turns out rather inspiring that, in analogy with the important role the symmetric matrix inequality takes in the proof of the well-known Simons inequality for closed minimal submanifolds in spheres (cf. [4, 17, 23]), the skew-symmetric matrix inequality also takes crucial effect in deducing a Simons-type inequality for Riemannian submersions. In order to state the result we first recall some notions about Riemannian submersions. The notions in Chapter 9 of the book [2] will be used throughout this paper.

Let M^{n+m} and B^n be (connected) Riemannian manifolds. A smooth map $\pi : M \rightarrow B$ is called a *Riemannian submersion* if π is of maximal rank and π_* preserves the lengths of horizontal vectors, *i.e.*, vectors orthogonal to the fibre $\pi^{-1}(b)$ for $b \in B$. Let \mathcal{V} denote the *vertical distribution* consisting of vertical vectors (tangent to the fibres) and \mathcal{H} denote the *horizontal distribution* consisting of horizontal vectors on M . The corresponding projections from TM to \mathcal{V} and \mathcal{H} are denoted by the same characters. For Riemannian submersions there are two fundamental tensors T and A on M defined by O’Neill [19] as follows. For vector fields E_1 and E_2 on M ,

$$(1.3) \quad \begin{aligned} T_{E_1} E_2 &:= \mathcal{H} D_{\mathcal{V} E_1} \mathcal{V} E_2 + \mathcal{V} D_{\mathcal{V} E_1} \mathcal{H} E_2, \\ A_{E_1} E_2 &:= \mathcal{H} D_{\mathcal{H} E_1} \mathcal{V} E_2 + \mathcal{V} D_{\mathcal{H} E_1} \mathcal{H} E_2, \end{aligned}$$

where D is the Levi-Civita connection on M . In fact, T is the second fundamental form along each fibre if it is restricted to vertical vectors, while A measures the obstruction to integrability of the horizontal distribution \mathcal{H} and hence it is called the *integrability tensor* of π . Moreover, some analogues of the Gauss-Codazzi equations for a Riemannian submersion obtained by O’Neill [19] are expressed in terms of T and A as well as their covariant derivatives. These equations will be recovered in Section 3 by moving frame method, which is an effective method firstly used to the study of Riemannian submersions though widely adopted in submanifold geometry. More details about T and A can be found in [2, 19]. Next we introduce the notion of *Yang-Mills* which has been intensely studied both in physics and in mathematics and also found important for Einstein Riemannian submersions (see for example [1, 2, 10] and references therein). Here we use the presentation given in [2]. Let X_1, \dots, X_n be a local orthonormal basis of the horizontal distribution \mathcal{H} . Define a co-differential operator $\check{\delta}$ over tensor fields on M by

$$\check{\delta} E := - \sum_{i=1}^n (D_{X_i} E)_{X_i}.$$

Then we say that \mathcal{H} satisfies the *Yang-Mills* condition if, for any vertical vector U and any horizontal vector X , we have

$$\langle \check{\delta}A(X), U \rangle - \langle A_X, T_U \rangle = 0,$$

where the bracket $\langle \cdot, \cdot \rangle$ denotes the metric of M and also its induced metric on tensors. As pointed out in [2], this condition depends only on \mathcal{H} and the metric of B and not on the family of metrics on the fibres. By properties of T and A , it is not hard to verify that when the fibres are totally geodesic, *i.e.*, $T = 0$, this condition is equivalent to

$$\check{\delta}A = 0,$$

which is one of the three sufficient and necessary conditions for M to be Einstein in this case. To be coherent with that in [2], we define the square norm of A by

$$(1.4) \quad |A|^2 := \sum_{i,j=1}^n \langle A_{X_i} X_j, A_{X_i} X_j \rangle = \sum_{i=1}^n \sum_{r=1}^m \langle A_{X_i} U_r, A_{X_i} U_r \rangle,$$

where $\{U_1, \dots, U_m\}$ is a local orthonormal basis of the vertical distribution \mathcal{V} . Besides several references cited in [2], it is noteworthy that the square norm of A has been also studied by Chen ([3], *etc.*) who denoted it by \check{A}_π and obtained its sharp upper bound for an arbitrary isometric immersion from M (with totally geodesic fibres) into a unit sphere in terms of square norm of the mean curvature of the immersion.

Now we are ready to state the main result as follows. For $x \in M$, we denote by $\check{\kappa}(x)$ the largest eigenvalue of the curvature operator $\check{R} : \bigwedge^2 TB \rightarrow \bigwedge^2 TB$ of B at $\pi(x) \in B$, $\check{\lambda}(x)$ the lowest eigenvalue of the Ricci curvature \check{r} of B at $\pi(x) \in B$ (thus $\check{\kappa}$, $\check{\lambda}$ are constant along any fibre), and $\hat{\mu}(x)$ the largest eigenvalue of the Ricci curvature \hat{r} of the fibre at x .

Theorem 1.2. *Let $\pi : M^{n+m} \rightarrow B^n$ be a Riemannian submersion with totally geodesic fibres and Yang-Mills horizontal distribution, *i.e.*, $T = 0$ and $\check{\delta}A = 0$. Suppose that M is closed. Then the following cases hold:*

(i) *If $n = 2$, then we have*

$$\int_M |A|^2 \hat{\mu} \, dV_M \geq 0;$$

(ii) *If $m = 1$, then we have*

$$\int_M |A|^2 (\check{\kappa} - \check{\lambda}) \, dV_M \geq 0;$$

(iii) *If $m \geq 2$ and $n = 3$, then we have*

$$\int_M |A|^2 \left(\frac{1}{6} |A|^2 + 2\hat{\mu} + \check{\kappa} - \check{\lambda} \right) \, dV_M \geq 0;$$

(iv) If $m \geq 2$ and $n \geq 4$, then we have

$$\int_M |A|^2 \left(\frac{1}{3} |A|^2 + 2\hat{\mu} + \check{\kappa} - \check{\lambda} \right) dV_M \geq 0.$$

Moreover, if $A \neq 0$, or equivalently, M is not locally a Riemannian product $B \times F$, then we have the following conclusions about the equality conditions:

- (a) In each case, if the equality holds, then each fibre has flat normal bundle in M and $|A|^2 \equiv \text{Const} =: C > 0$, which implies further the following:
 - (a1) In case (i), $\hat{\mu} \equiv 0$;
 - (a2) In case (ii), $\check{\kappa} - \check{\lambda} \equiv 0$;
 - (a3) In case (iii), $\hat{\mu} \equiv \frac{1}{12}C$, $\check{\kappa} - \check{\lambda} \equiv \frac{-1}{3}C$;
 - (a4) In case (iv), $\hat{\mu} \equiv \frac{1}{6}C$, $\check{\kappa} - \check{\lambda} \equiv \frac{-2}{3}C$.
- (b) If the equality in (iii) or (iv) holds, then $m \geq 3$ and at each point of M there exist an orthonormal vertical basis $\{U_1, \dots, U_m\}$ and an orthonormal horizontal basis $\{X_1, \dots, X_n\}$ such that the $(n \times n)$ skew-symmetric matrices

$$A^r := \left(\langle A_{X_i} U_r, X_j \rangle \right)_{n \times n}, \quad r = 1, \dots, m,$$

are in the forms of the matrices in the equality conditions of (i) or (ii) of Theorem 1.1 respectively. Furthermore, under these basis, the following decompositions hold

$$\begin{aligned} \hat{r} &= \hat{\mu} I_3 \oplus \hat{r}', \\ \check{R} &\equiv \check{\kappa} I_3, \quad \check{r} \equiv 2\check{\kappa} I_3, \quad \text{in case (iii),} \\ \check{R} &= \check{\kappa} I_6 \oplus \check{R}', \quad \check{r} \equiv \check{\lambda} I_4 \oplus \check{r}', \quad \text{in case (iv),} \end{aligned}$$

where $\hat{r}' = \hat{r}|_{\text{span}\{U_4, \dots, U_m\}}$, $\check{R}' = \check{R}|_{\text{span}\{X_i \wedge X_j | 1 \leq i \leq n, 5 \leq j \leq n\}}$, and $\check{r}' = \check{r}|_{\text{span}\{X_5, \dots, X_n\}}$. In particular, when $m = 3$, the fibres have constant sectional curvature. Similarly, when $3 \leq n \leq 5$, the base manifold B^n has constant sectional curvature. More precisely and specifically, we have the following (c-d).

- (c) When $m = 3$, if the equality in (iii) holds, then there exist some $a > 0$ such that
 - (c1) all fibres are isometric to a manifold F^3 of constant sectional curvature a ;
 - (c2) the base manifold B^3 has constant sectional curvature $8a$;
 - (c3) the following identities hold:

$$|A|^2 \equiv 24a,$$

$$K_{rs} \equiv a, \quad K_{ij} \equiv -4a, \quad K_{ir} = \begin{cases} 0 & \text{for } (i, r) = (1, 3), (2, 2), (3, 1) \\ 4a & \text{otherwise,} \end{cases}$$

$$R_{rs} \equiv 10a\delta_{rs}, \quad R_{ij} \equiv 0, \quad R_{ir} \equiv 0,$$

- where K_{rs}, K_{ij}, K_{ir} (resp. R_{rs}, R_{ij}, R_{ir}) are sectional curvatures (resp. Ricci curvatures) of M on the 2-planes spanned by $\{U_r, U_s\}, \{X_i, X_j\}, \{X_i, U_r\}$, respectively, under the basis $\{U_r\}$ and $\{X_i\}$ given in case (b).
- (d) When $m = 3$, if the equality in (iv) holds, then there exist some $a > 0$ such that all fibres are isometric to a manifold F^3 of constant sectional curvature a . In addition,
- (d1) if $n = 4$, then the submersion π is covered by the Hopf fibration $\pi_0 : S^7(\frac{1}{\sqrt{a}}) \rightarrow S^4(\frac{1}{2\sqrt{a}})$, i.e., there are two covering maps $\pi_1 : S^7(\frac{1}{\sqrt{a}}) \rightarrow M^7$ and $\pi_2 : S^4(\frac{1}{2\sqrt{a}}) \rightarrow B^4$ such that $\pi_2 \circ \pi_0 = \pi \circ \pi_1$;
- (d2) if $n = 5$, then the base manifold B^5 has constant sectional curvature $\frac{8}{3}a$, and the following identities hold (with the same notations as in (c3)):

$$|A|^2 \equiv 12a,$$

$$K_{rs} \equiv a, \quad K_{ij} = \begin{cases} \frac{-1}{3}a & \text{for } 1 \leq i < j \leq 4 \\ \frac{8}{3}a & \text{for } 1 \leq i < j = 5, \end{cases} \quad K_{ir} = \begin{cases} a & \text{for } 1 \leq i \leq 4 \\ 0 & \text{for } i = 5, \end{cases}$$

$$R_{rs} \equiv 6a\delta_{rs}, \quad R_{ij} = \begin{cases} \frac{14}{3}a\delta_{ij} & \text{for } 1 \leq i, j \leq 4 \\ \frac{32}{3}a\delta_{ij} & \text{for } 1 \leq i \leq j = 5, \end{cases} \quad R_{ir} \equiv 0.$$

Remark 1.1. As we mentioned previously, the Yang-Mills condition is implied by the Einstein condition of M when the fibres are totally geodesic. Therefore, examples satisfying our assumptions of the theorem are plentiful (cf. [2]). Note that the corresponding pointwise inequalities with the same equality conclusions also hold when M is not closed provided that $|A|^2$ is constant on M , which is also implied by the Einstein condition of M .

Remark 1.2. Besides the classification problem, searching examples of Riemannian submersions in (c) and (d2) of the theorem might make sense to itself. For instance, the base manifold B^3 can not be simply connected in case (c) due to the facts that any principal G -bundle over S^3 is trivial if G is a Lie group (in which case $A \equiv 0$) and that any Riemannian submersion from a complete manifold M with totally geodesic fibres is a fibre bundle associated to a principal G -bundle for some Lie group G (cf. Remark 9.57 in [2], and [15, 18]). Therefore, searching examples in (c) should start with a non-simply-connected 3-dimensional base manifold of constant sectional curvature.

To conclude the introduction, we remark that as the Chern conjecture, the classification of the equality case and Peng-Terng pinching theorems based on the Simons inequality in submanifold geometry (cf. [4, 5, 9, 16, 17, 20, 21], etc.), one can now ask the “dual” version for Riemannian submersions with square norm of the integrability tensor A instead of square norm of the second fundamental form.

2. DDVV-TYPE SKEW-SYMMETRIC MATRIX INEQUALITY

2.1. Notations and preparing lemmas. Throughout this section, we denote by $M(m, n)$ the space of $m \times n$ real matrices, $M(n)$ the space of $n \times n$ real matrices and $\mathfrak{o}(n)$ the $N := \frac{n(n-1)}{2}$ dimensional subspace of skew-symmetric matrices in $M(n)$.

For every (i, j) with $1 \leq i < j \leq n$, let $\tilde{E}_{ij} := \frac{1}{\sqrt{2}}(E_{ij} - E_{ji})$, where $E_{ij} \in M(n)$ is the matrix with (i, j) entry 1 and all others 0. Clearly $\{\tilde{E}_{ij}\}_{i < j}$ is an orthonormal basis of $\mathfrak{o}(n)$. Let us take an order of the indices set $S := \{(i, j) | 1 \leq i < j \leq n\}$ by

$$(2.1) \quad (i, j) < (k, l) \text{ if and only if } i < k \text{ or } i = k < j < l.$$

In this way we can identify S with $\{1, \dots, N\}$ and write elements of S in Greek, i.e. for $\alpha = (i, j) \in S$, we can say $1 \leq \alpha \leq N$.

For $\alpha = (i, j) < (k, l) = \beta$ in S , direct calculations show that

$$(2.2) \quad \|[\tilde{E}_\alpha, \tilde{E}_\beta]\|^2 = \begin{cases} \frac{1}{2}, & i < j = k < l \text{ or } i = k < j < l \text{ or } i < k < j = l; \\ 0, & \text{otherwise,} \end{cases}$$

and for any $\alpha, \beta \in S$,

$$(2.3) \quad \sum_{\gamma \in S} \langle [\tilde{E}_\alpha, \tilde{E}_\gamma], [\tilde{E}_\beta, \tilde{E}_\gamma] \rangle = (n-2)\delta_{\alpha\beta},$$

where $\delta_{\alpha\beta} = \delta_{ik}\delta_{jl}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product of $M(n)$.

Let $\{\tilde{Q}_\alpha\}_{\alpha \in S}$ be any orthonormal basis of $\mathfrak{o}(n)$. There exists a unique orthogonal matrix $Q \in O(N)$ such that $(\tilde{Q}_1, \dots, \tilde{Q}_N) = (\tilde{E}_1, \dots, \tilde{E}_N)Q$, i.e. $\tilde{Q}_\alpha = \sum_{\beta} q_{\beta\alpha} \tilde{E}_\beta$ for $Q = (q_{\alpha\beta})_{N \times N}$. If we set $\tilde{Q}_\alpha = (\tilde{q}_{ij}^\alpha)_{n \times n}$, then $\tilde{q}_{ij}^\alpha = -\tilde{q}_{ji}^\alpha = \frac{1}{\sqrt{2}}q_{\beta\alpha}$ for $\beta = (i, j) \in S$. Henceforth, this correspondence between an orthonormal basis $\{\tilde{Q}_\alpha\}_{\alpha \in S}$ of $\mathfrak{o}(n)$ and an orthogonal matrix $Q \in O(N)$ is regarded known.

Let $\lambda_1, \dots, \lambda_{[\frac{n}{2}]}$ be $[\frac{n}{2}]$ real numbers satisfying $\sum_i \lambda_i^2 = \frac{1}{2}$ and $\lambda_1 \geq \dots \geq \lambda_{[\frac{n}{2}]} \geq 0$. Denote by $I := \{(i, j) \in S | (\lambda_i + \lambda_j)^2 > \frac{2}{3}\}$ and n_0 the number of elements of I . It is easily seen that $n_0 = 0$ when $n = 3$. Moreover, we have

Lemma 2.1. *If I is not empty, i.e. $n_0 \geq 1$, then*

$$I = \{1\} \times \{2, \dots, n_0 + 1\}, \quad n_0 + 1 \leq [\frac{n}{2}].$$

Proof. Obviously, by the assumptions of λ_i 's, $(1, 2) \in I$ if I is not empty. It suffices to prove that $(2, 3)$ is not in I . Otherwise, we have

$$(\lambda_1 + \lambda_2)^2 \geq (\lambda_1 + \lambda_3)^2 \geq (\lambda_2 + \lambda_3)^2 > \frac{2}{3},$$

and thus

$$4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \geq (\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 > 2,$$

which contradicts with $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq \sum_i \lambda_i^2 = \frac{1}{2}$. \square

Lemma 2.2. *We have*

$$\sum_{(i,j) \in I} \left((\lambda_i + \lambda_j)^2 - \frac{2}{3} \right) \leq \frac{1}{3},$$

where the equality holds if and only if $n_0 = 1$, $\lambda_1 = \lambda_2 = \frac{1}{2}$ and all other λ_j 's 0.

Proof. By Lemma 2.1,

$$\begin{aligned} \sum_{(i,j) \in I} \left[(\lambda_i + \lambda_j)^2 - \frac{2}{3} \right] &= \sum_{j=2}^{n_0+1} (\lambda_1^2 + \lambda_j^2 + 2\lambda_1\lambda_j) - \frac{2}{3}n_0 \\ &= n_0\lambda_1^2 + \sum_{j=2}^{n_0+1} \lambda_j^2 + 2\lambda_1 \sum_{j=2}^{n_0+1} \lambda_j - \frac{2}{3}n_0 \\ &\leq (n_0 + 1)\lambda_1^2 + \sum_{j=2}^{n_0+1} \lambda_j^2 + \left(\sum_{j=2}^{n_0+1} \lambda_j \right)^2 - \frac{2}{3}n_0 \\ &\leq (n_0 + 1) \left(\lambda_1^2 + \sum_{j=2}^{n_0+1} \lambda_j^2 \right) - \frac{2}{3}n_0 \\ &\leq (n_0 + 1) \sum_i \lambda_i^2 - \frac{2}{3}n_0 = \frac{n_0 + 1}{2} - \frac{2}{3}n_0 \leq \frac{1}{3}, \end{aligned}$$

where the equality condition is easily seen from the proof. \square

Lemma 2.3. *For any $Q \in O(N)$, $\alpha \in S$ and any subset $J_\alpha \subset S$, we have*

$$\sum_{\beta \in J_\alpha} \left(\|\tilde{Q}_\alpha, \tilde{Q}_\beta\|^2 - \frac{2}{3} \right) \leq \frac{2}{3}.$$

Proof. Given $\alpha \in S$, under some $O(n) \subset K$ action, without loss of generality, we can assume

$$\tilde{Q}_\alpha = \text{diag} \left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_{[\frac{n}{2}]} \\ -\lambda_{[\frac{n}{2}]} & 0 \end{pmatrix}, 0 \right),$$

where $\lambda_1 \geq \dots \geq \lambda_{[\frac{n}{2}]} \geq 0$, $\sum_i \lambda_i^2 = \frac{1}{2}$ and the last 0 exists only if n is odd.

Put

$$(2.4) \quad U := \text{diag} \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{-1}}{\sqrt{2}} \\ \frac{\sqrt{-1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \dots, \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{-1}}{\sqrt{2}} \\ \frac{\sqrt{-1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, 1 \right),$$

where the last 1 exists only if n is odd. Set $\tilde{Q}_\gamma := U\sqrt{-1}\tilde{Q}_\gamma U^* := (\tilde{q}_{ij}^\gamma)$ for $\gamma \in S$, where U^* denotes the conjugate transpose. Then the following identities can be easily

verified for $k, l = 1, \dots, [\frac{n}{2}]$ and $k < l$:

$$\begin{aligned}
\tilde{q}_{2k-1,2k-1}^\gamma &= -\tilde{q}_{2k,2k}^\gamma = \tilde{q}_{2k-1,2k}^\gamma, \quad \tilde{q}_{n,n}^\gamma = 0 \quad \text{if } n \text{ is odd}; \\
\tilde{q}_{2k-1,2k}^\gamma &= \tilde{q}_{2k,2k-1}^\gamma = 0; \\
\tilde{q}_{2k-1,2l-1}^\gamma &= -\overline{\tilde{q}_{2k,2l}^\gamma} = \frac{1}{2} \{ (\tilde{q}_{2k-1,2l}^\gamma - \tilde{q}_{2k,2l-1}^\gamma) + \sqrt{-1}(\tilde{q}_{2k-1,2l-1}^\gamma + \tilde{q}_{2k,2l}^\gamma) \}; \\
\tilde{q}_{2k-1,2l}^\gamma &= -\overline{\tilde{q}_{2k,2l-1}^\gamma} = \frac{1}{2} \{ (\tilde{q}_{2k-1,2l-1}^\gamma - \tilde{q}_{2k,2l}^\gamma) + \sqrt{-1}(\tilde{q}_{2k,2l-1}^\gamma + \tilde{q}_{2k-1,2l}^\gamma) \}; \\
\tilde{q}_{2k-1,n}^\gamma &= \sqrt{-1}\overline{\tilde{q}_{2k,n}^\gamma} = \overline{\tilde{q}_{n,2k-1}^\gamma} = -\sqrt{-1}\tilde{q}_{n,2k}^\gamma = \frac{1}{\sqrt{2}}(-\tilde{q}_{2k,n}^\gamma + \sqrt{-1}\tilde{q}_{2k-1,n}^\gamma) \text{ if } n \text{ is odd}.
\end{aligned}$$

In particular,

$$\check{Q}_\alpha = \text{diag}(\lambda_1, -\lambda_1, \dots, \lambda_{[\frac{n}{2}]}, -\lambda_{[\frac{n}{2}]}, 0) =: \text{diag}(u_1, u_2, \dots, u_n).$$

For any $(i, j) \in \acute{S} := \{(i, j) \in S \mid (i, j) \neq (2k-1, 2k), 1 \leq k \leq [\frac{n}{2}]\}$, it follows from the identities above that

$$\sum_{\gamma \in S} |\tilde{q}_{ij}^\gamma|^2 = \frac{1}{2}.$$

As for the proof, we take $(2k-1, 2l-1) \in \acute{S}$ for example as the following:

$$\begin{aligned}
\sum_{\gamma \in S} |\tilde{q}_{2k-1,2l-1}^\gamma|^2 &= \sum_{\gamma \in S} \frac{1}{4} \left((\tilde{q}_{2k-1,2l}^\gamma)^2 + (\tilde{q}_{2k,2l-1}^\gamma)^2 + (\tilde{q}_{2k-1,2l-1}^\gamma)^2 + (\tilde{q}_{2k,2l}^\gamma)^2 \right. \\
&\quad \left. - 2\tilde{q}_{2k-1,2l}^\gamma \tilde{q}_{2k,2l-1}^\gamma + 2\tilde{q}_{2k-1,2l-1}^\gamma \tilde{q}_{2k,2l}^\gamma \right) \\
&= \sum_{\gamma \in S} \frac{1}{8} \left((q_{(2k-1,2l)\gamma})^2 + (q_{(2k,2l-1)\gamma})^2 + (q_{(2k-1,2l-1)\gamma})^2 + (q_{(2k,2l)\gamma})^2 \right. \\
&\quad \left. - 2q_{(2k-1,2l)\gamma} q_{(2k,2l-1)\gamma} + 2q_{(2k-1,2l-1)\gamma} q_{(2k,2l)\gamma} \right) \\
&= \frac{1}{8} (1 + 1 + 1 + 1 + 0 + 0) = \frac{1}{2}.
\end{aligned}$$

Denote by $\check{S} := \{(i, j) \in \acute{S} \mid (u_i - u_j)^2 > \frac{2}{3}\}$. Since $\sum_i \lambda_i^2 = \frac{1}{2}$, we find that $u_i u_j < 0$ for $(i, j) \in \check{S}$ and hence $(u_i, u_j) = (\lambda_k, -\lambda_l)$ or $(-\lambda_k, \lambda_l)$ for some $(k, l) \in I$. Then by the

preceding identities and Lemma 2.2, we complete the proof of the lemma as follows:

$$\begin{aligned}
\sum_{\beta \in J_\alpha} \left(\|\tilde{Q}_\alpha, \tilde{Q}_\beta\|^2 - \frac{2}{3} \right) &= \sum_{\beta \in J_\alpha} \left(\|\tilde{Q}_\alpha, \tilde{Q}_\beta\|^2 - \frac{2}{3} \right) \\
&= \sum_{\beta \in J_\alpha} \sum_{i,j=1}^n \left((u_i - u_j)^2 - \frac{2}{3} \right) |\tilde{q}_{ij}^\beta|^2 \\
&\leq \sum_{\beta \in J_\alpha} 2 \sum_{i < j} \left((u_i - u_j)^2 - \frac{2}{3} \right) |\tilde{q}_{ij}^\beta|^2 \\
&= 2 \sum_{\beta \in J_\alpha} \sum_{(i,j) \in \check{S}} \left((u_i - u_j)^2 - \frac{2}{3} \right) |\tilde{q}_{ij}^\beta|^2 \\
&\leq 2 \sum_{(i,j) \in \check{S}} \left((u_i - u_j)^2 - \frac{2}{3} \right) \sum_{\beta \in J_\alpha} |\tilde{q}_{ij}^\beta|^2 \\
&\leq 2 \sum_{(i,j) \in \check{S}} \left((u_i - u_j)^2 - \frac{2}{3} \right) \sum_{\beta \in S} |\tilde{q}_{ij}^\beta|^2 \\
&\leq 4 \sum_{(k,l) \in I} \left((\lambda_k + \lambda_l)^2 - \frac{2}{3} \right) \frac{1}{2} \\
&\leq \frac{2}{3}.
\end{aligned}$$

□

Lemma 2.4. For any $Q \in O(N)$ and $\alpha \in S$, we have

$$\sum_{\beta \in S} \|\tilde{Q}_\alpha, \tilde{Q}_\beta\|^2 = n - 2.$$

Proof. It follows from (2.3) that

$$\begin{aligned}
\sum_{\beta \in S} \|\tilde{Q}_\alpha, \tilde{Q}_\beta\|^2 &= \sum_{\beta \in \gamma\tau\xi\eta} q_{\gamma\alpha} q_{\xi\alpha} q_{\tau\beta} q_{\eta\beta} \langle [\tilde{E}_\gamma, \tilde{E}_\tau], [\tilde{E}_\xi, \tilde{E}_\eta] \rangle \\
&= \sum_{\gamma\tau\xi\eta} q_{\gamma\alpha} q_{\xi\alpha} \delta_{\tau\eta} \langle [\tilde{E}_\gamma, \tilde{E}_\tau], [\tilde{E}_\xi, \tilde{E}_\eta] \rangle \\
&= \sum_{\gamma\xi} q_{\gamma\alpha} q_{\xi\alpha} \sum_{\tau} \langle [\tilde{E}_\gamma, \tilde{E}_\tau], [\tilde{E}_\xi, \tilde{E}_\tau] \rangle \\
&= \sum_{\gamma\xi} q_{\gamma\alpha} q_{\xi\alpha} (n-2) \delta_{\gamma\xi} = (n-2) \sum_{\gamma} q_{\gamma\alpha}^2 = n-2.
\end{aligned}$$

□

Lemma 2.5. Let A, B be $(n \times n)$ real skew-symmetric matrices.

(i) If $n = 3$, then we have

$$\|[A, B]\|^2 \leq \frac{1}{2} \|A\|^2 \|B\|^2,$$

where the equality holds if and only if there is a $P \in O(3)$ such that

$$PAP^t = C_1, \quad PBP^t = aC_2 + bC_3,$$

where C_1, C_2, C_3 are the matrices in Theorem 1.1 and a, b are real numbers.

(ii) If $n \geq 4$, then we have

$$\|[A, B]\|^2 \leq \|A\|^2 \|B\|^2,$$

where the equality holds if and only if there is a $P \in O(n)$ such that

$$PAP^t = \text{diag}(D_1, 0), \quad PBP^t = a \cdot \text{diag}(D_2, 0) + b \cdot \text{diag}(D_3, 0),$$

where D_1, D_2, D_3 are the matrices in Theorem 1.1 and a, b are real numbers.

Proof. (i) As A is now a (3×3) real skew-symmetric matrix, there is a $P \in O(3)$ such that

$$PAP^t = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = C_1.$$

Denote by $PBP^t := (b_{ij}) \in \mathfrak{o}(3)$. Then direct computation shows that

$$[PAP^t, PBP^t] = \begin{pmatrix} 0 & 0 & \lambda b_{23} \\ 0 & 0 & -\lambda b_{13} \\ -\lambda b_{23} & \lambda b_{13} & 0 \end{pmatrix}.$$

Thus

$$\|[A, B]\|^2 = \|[PAP^t, PBP^t]\|^2 = 2\lambda^2(b_{23}^2 + b_{13}^2) \leq \frac{1}{2}\|A\|^2\|B\|^2,$$

where the equality holds if and only if $b_{12} = 0$, i.e., PBP^t lies in $\text{Span}\{C_2, C_3\}$.

(ii) As A is now a $(n \times n)$ real skew-symmetric matrix, there is a $P \in O(n)$ such that

$$PAP^t = \text{diag} \left(\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \lambda_{[\frac{n}{2}]} \\ -\lambda_{[\frac{n}{2}]} & 0 \end{pmatrix}, 0 \right),$$

where $\lambda_1 \geq \dots \geq \lambda_{[\frac{n}{2}]} \geq 0$ and the last 0 exists only if $n \geq 4$ is odd.

Let U be the unitary matrix defined in (2.4). Then we have

$$\check{A} := U\sqrt{-1}PAP^tU^* = \text{diag}(\lambda_1, -\lambda_1, \dots, \lambda_{[\frac{n}{2}]}, -\lambda_{[\frac{n}{2}]}, 0) := \text{diag}(u_1, u_2, \dots, u_n).$$

Put

$$\check{B} := U\sqrt{-1}PBP^tU^* := (b_{ij}), \quad \text{sgn}(n) = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Then it follows from the proof of Lemma 2.3 that $b_{2k-1,2k} = 0$ and

$$\begin{aligned}
\| [A, B] \|^2 &= \| [\check{A}, \check{B}] \|^2 = \sum_{i,j=1}^n (u_i - u_j)^2 |b_{ij}|^2 \\
&= 2 \left(\sum_{k < l} [(\lambda_k - \lambda_l)^2 (|b_{2k-1,2l-1}|^2 + |b_{2k,2l}|^2) + (\lambda_k + \lambda_l)^2 (|b_{2k-1,2l}|^2 + |b_{2k,2l-1}|^2)] \right) \\
&\quad + 2(\operatorname{sgn}(n)) \sum_k \lambda_k^2 (|b_{2k-1,n}|^2 + |b_{2k,n}|^2) \\
&\leq 2 \left(\sum_{k < l} (\lambda_1 + \lambda_2)^2 (|b_{2k-1,2l-1}|^2 + |b_{2k,2l}|^2 + |b_{2k-1,2l}|^2 + |b_{2k,2l-1}|^2) \right) \\
&\quad + 2(\operatorname{sgn}(n)) \sum_k \lambda_1^2 (|b_{2k-1,n}|^2 + |b_{2k,n}|^2) \\
&\leq 2 \|A\|^2 \left(\sum_{k < l} (|b_{2k-1,2l-1}|^2 + |b_{2k,2l}|^2 + |b_{2k-1,2l}|^2 + |b_{2k,2l-1}|^2) \right) \\
&\quad + \|A\|^2 (\operatorname{sgn}(n)) \sum_k (|b_{2k-1,n}|^2 + |b_{2k,n}|^2) \\
&\leq \|A\|^2 \|B\|^2.
\end{aligned}$$

Analyzing these inequalities, we find that the equality in this case holds if and only if $\lambda_1 = \lambda_2 = \frac{1}{2} \|A\|$, $\lambda_j = 0$ for $j > 2$, and all b_{ij} 's are zero except $b_{14} = \bar{b}_{41}$ and $b_{23} = \bar{b}_{32}$, which is equivalent to that PAP^t, PBP^t are in the forms specified in the lemma. \square

Now let $\varphi : M(m, n) \longrightarrow M(C_m^2, C_n^2)$ be the map defined by $\varphi(A)_{(i,j)(k,l)} := A_{ij}^{kl}$, where $C_m^2 = \frac{m(m-1)}{2}$, $1 \leq i < j \leq m$, $1 \leq k < l \leq n$ and $A_{ij}^{kl} = a_{ik}a_{jl} - a_{il}a_{jk}$ is the determinant of the sub-matrix of $A := (a_{ij})$ with the rows i, j , the columns k, l , arranged with the same ordering as in (2.1). It is easily seen that $\varphi(I_n) = I_{C_n^2}$ (preserving identity matrices), $\varphi(A)^t = \varphi(A^t)$ and the following

Lemma 2.6. *The map φ preserves the matrix product, i.e. $\varphi(AB) = \varphi(A)\varphi(B)$ holds for $A \in M(m, k)$, $B \in M(k, n)$.*

We will also need the following exercise of linear algebra in the proof of the equality case of Theorem 1.1.

Lemma 2.7. *Let A, B be two matrices in $M(m, n)$. Then $AA^t = BB^t$ if and only if $A = BR$ for some $R \in O(n)$.*

2.2. Proof of Theorem 1.1. Let B_1, \dots, B_m be any $(n \times n)$ real skew-symmetric matrices. Their coefficients under the standard basis $\{\tilde{E}_\alpha\}_{\alpha \in S}$ of $\mathfrak{o}(n)$ are determined by a matrix $B \in M(N, m)$ as $(B_1, \dots, B_m) = (\tilde{E}_1, \dots, \tilde{E}_N)B$. Taking the same ordering as in (2.1) for $1 \leq r < s \leq m$ and $1 \leq \alpha < \beta \leq N$, we arrange $\{[B_r, B_s]\}_{r < s}$,

$\{[\tilde{E}_\alpha, \tilde{E}_\beta]\}_{\alpha < \beta}$ into C_m^2 , C_N^2 -dimensional vectors respectively. We first observe that

$$([B_1, B_2], \dots, [B_{m-1}, B_m]) = ([\tilde{E}_1, \tilde{E}_2], \dots, [\tilde{E}_{N-1}, \tilde{E}_N]) \cdot \varphi(B).$$

Let $C(\tilde{E})$ denote the matrix in $M(C_N^2)$ defined by $C(\tilde{E})_{(\alpha, \beta)(\gamma, \tau)} := \langle [\tilde{E}_\alpha, \tilde{E}_\beta], [\tilde{E}_\gamma, \tilde{E}_\tau] \rangle$, for $1 \leq \alpha < \beta \leq N$, $1 \leq \gamma < \tau \leq N$. Moreover we will use the same notation for $\{B_r\}$ and $\{\tilde{Q}_\alpha\}$, i.e., $C(B)$ and $C(Q)$ respectively. Then it is obvious that

$$C(B) = \varphi(B^t)C(\tilde{E})\varphi(B), \quad C(Q) = \varphi(Q^t)C(\tilde{E})\varphi(Q).$$

Since BB^t is a $(N \times N)$ semi-positive definite matrix, there exists an orthogonal matrix $Q \in SO(N)$ such that $BB^t = Q \operatorname{diag}(x_1, \dots, x_N) Q^t$ with $x_\alpha \geq 0$, $1 \leq \alpha \leq N$. Thus

$$\sum_{r=1}^m \|B_r\|^2 = \|B\|^2 = \sum_{\alpha=1}^N x_\alpha$$

and hence by Lemma 2.6,

$$\begin{aligned} \sum_{r,s=1}^m \|[B_r, B_s]\|^2 &= 2Tr \, C(B) = 2Tr \, \varphi(B^t)C(\tilde{E})\varphi(B) = 2Tr \, \varphi(BB^t)C(\tilde{E}) \\ &= 2Tr \, \varphi(\operatorname{diag}(x_1, \dots, x_N))C(Q) = \sum_{\alpha, \beta=1}^N x_\alpha x_\beta \|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2. \end{aligned}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Put $d(n) := \frac{1}{3}$ if $n = 3$ and $\frac{2}{3}$ if $n \geq 4$. It follows from the arguments above that the inequalities of the theorem are equivalent to the following

$$(2.5) \quad \sum_{\alpha, \beta=1}^N x_\alpha x_\beta \|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 \leq d(n) \left(\sum_{\alpha=1}^N x_\alpha \right)^2, \text{ for any } x \in \mathbb{R}_+^N, \, Q \in SO(N),$$

where $\mathbb{R}_+^N := \{0 \neq x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_\alpha \geq 0, 1 \leq \alpha \leq N\}$.

For $n = 3$, $N = \frac{n(n-1)}{2} = 3$ and by Lemma 2.5, we have $\|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 \leq \frac{1}{2}$ and thus

$$\sum_{\beta \in S} \|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 \leq \frac{1}{2} \times 2 = 1.$$

On the other hand, it follows from Lemma 2.4 that $\sum_{\beta \in S} \|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 = n - 2 = 1$. Therefore, we get

$$\|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 = \frac{1}{2}, \text{ for any } \alpha \neq \beta \in S.$$

In fact, this equality just says that the cross product of two orthogonal unit vectors in \mathbb{R}^3 is still a unit vector if we identify $\mathfrak{o}(3)$ with \mathbb{R}^3 and correspond the commutator operator to the cross product. So in this case, the inequality (2.5) is equivalent to

$$x_1 x_2 + x_2 x_3 + x_3 x_1 \leq \frac{1}{3} (x_1 + x_2 + x_3)^2, \text{ for any } x \in \mathbb{R}_+^3,$$

which is easily verified by

$$x_1x_2 + x_2x_3 + x_3x_1 - \frac{1}{3}(x_1 + x_2 + x_3)^2 = -\frac{1}{6}\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right) \leq 0.$$

Note that the equality above holds if and only if $x_1 = x_2 = x_3 := \lambda^2$, i.e., $BB^t = \lambda^2 I_3$, which, by Lemma 2.7, is equivalent to that there is a $R \in O(m)$ such that

$$(B_1, \dots, B_m) = (\tilde{E}_{12}, \tilde{E}_{13}, \tilde{E}_{23}) \cdot \left(\lambda I_3, 0_{3 \times (m-3)} \right) R = (C_1, C_2, C_3, 0, \dots, 0)R.$$

This completes the proof of (i) of Theorem 1.1.

Now we consider the case (ii). Put

$$f_Q(x) = F(x, Q) := \sum_{\alpha, \beta=1}^N x_\alpha x_\beta \|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2 - \frac{2}{3} \left(\sum_{\alpha=1}^N x_\alpha \right)^2.$$

Then F is a continuous function defined on $\mathbb{R}^N \times SO(N)$ and thus uniformly continuous on any compact subset of $\mathbb{R}^N \times SO(N)$. Let $\Delta := \{x \in \mathbb{R}_+^N \mid \sum_{\alpha} x_\alpha = 1\}$ and for any sufficiently small $\varepsilon > 0$, $\Delta_\varepsilon := \{x \in \Delta \mid x_\alpha \geq \varepsilon, 1 \leq \alpha \leq N\}$. Also let

$$G := \{Q \in SO(N) \mid f_Q(x) \leq 0, \text{ for all } x \in \Delta\},$$

$$G_\varepsilon := \{Q \in SO(N) \mid f_Q(x) < 0, \text{ for all } x \in \Delta_\varepsilon\}.$$

We claim that $G = \lim_{\varepsilon \rightarrow 0} G_\varepsilon = SO(N)$. Note that this implies (2.5) and thus proves the inequality. In fact we can show

$$(2.6) \quad G_\varepsilon = SO(N) \text{ for any sufficiently small } \varepsilon > 0.$$

To prove (2.6), we use the continuity method, in which we must prove the following three properties:

- (a) $I_N \in G_\varepsilon$ (and thus $G_\varepsilon \neq \emptyset$);
- (b) G_ε is open in $SO(N)$;
- (c) G_ε is closed in $SO(N)$.

Since F is uniformly continuous on $\Delta_\varepsilon \times SO(N)$, (b) is obvious.

Proof of (a). For any $x \in \Delta_\varepsilon$, $f_{I_N}(x) = \sum_{\alpha, \beta=1}^N x_\alpha x_\beta \|[\tilde{E}_\alpha, \tilde{E}_\beta]\|^2 - \frac{2}{3} \left(\sum_{\alpha=1}^N x_\alpha \right)^2$.

It follows from (2.2) that

$$\begin{aligned} f_{I_N}(x) &= \sum_{i < j < k} (x_{ij}x_{jk} + x_{ij}x_{ik} + x_{ik}x_{jk}) - \frac{2}{3} \left(\sum_{i < j} x_{ij} \right)^2 \\ &< \sum_{i < j < k} (x_{ij}x_{jk} + x_{ij}x_{ik} + x_{ik}x_{jk}) - \frac{2}{3} \sum_{i < j < k} 2(x_{ij}x_{jk} + x_{ij}x_{ik} + x_{ik}x_{jk}) \\ &< 0, \end{aligned}$$

which means $I_N \in G_\varepsilon$. □

Proof of (c). We only need to prove the following a priori estimate: Suppose $f_Q(x) \leq 0$ for every $x \in \Delta_\varepsilon$. Then $f_Q(x) < 0$ for every $x \in \Delta_\varepsilon$.

The proof of this estimate is as follows: If there is a point $y \in \Delta_\varepsilon$ such that $f_Q(y) = 0$, we can assume without loss of generality that

$$y \in \Delta_\varepsilon^\gamma := \{x \in \Delta_\varepsilon \mid x_\alpha > \varepsilon \text{ for } \alpha \leq \gamma \text{ and } x_\beta = \varepsilon \text{ for } \beta > \gamma\}$$

for some $1 \leq \gamma \leq N$. Then y is a maximum point of $f_Q(x)$ in the cone spanned by Δ_ε and an interior maximum point in $\Delta_\varepsilon^\gamma$. Hence there exist numbers $b_{\gamma+1}, \dots, b_N$ and a number a such that

$$(2.7) \quad \begin{cases} \left(\frac{\partial f_Q}{\partial x_1}(y), \dots, \frac{\partial f_Q}{\partial x_\gamma}(y) \right) = 2a(1, \dots, 1), \\ \left(\frac{\partial f_Q}{\partial x_{\gamma+1}}(y), \dots, \frac{\partial f_Q}{\partial x_N}(y) \right) = 2(b_{\gamma+1}, \dots, b_N) \end{cases}$$

or equivalently

$$(2.8) \quad \sum_{\beta=1}^N y_\beta (\|[\tilde{Q}_\alpha, \tilde{Q}_\beta]\|^2) - \frac{2}{3} = \begin{cases} a & \alpha \leq \gamma, \\ b_\alpha & \alpha > \gamma. \end{cases}$$

Hence

$$f_Q(y) = \left(\sum_{\alpha=1}^{\gamma} y_\alpha \right) a + \left(\sum_{\alpha=\gamma+1}^N b_\alpha \right) \varepsilon = 0 \quad \text{and} \quad \sum_{\alpha=1}^{\gamma} y_\alpha + (N - \gamma)\varepsilon = 1.$$

Meanwhile, we see $\frac{\partial f_Q}{\partial \nu}(y) = 2(a\gamma + \sum_{\alpha=\gamma+1}^N b_\alpha) \leq 0$, where $\nu = (1, \dots, 1)$ is the vector normal to Δ in \mathbb{R}^N . For any sufficiently small ε (such as $\varepsilon < 1/N$), it follows from the above three formulas that $a \geq 0$. Without loss of generality, we assume $y_1 = \max\{y_1, \dots, y_\gamma\} > \varepsilon$. Let $J := \{\beta \in S \mid \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 \geq \frac{2}{3}\}$, and let n_1 be the number of elements of J . Now combining Lemma 2.3, Lemma 2.4 and Equation (2.8) will give a contradiction as follows:

$$(2.9) \quad \begin{aligned} \frac{2}{3} &\leq \frac{2}{3} + a = \sum_{\beta=2}^N y_\beta \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 \\ &= \sum_{\beta \in J} y_\beta (\|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 - \frac{2}{3}) + \frac{2}{3} \sum_{\beta \in J} y_\beta + \sum_{\beta \in S/J} y_\beta \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 \\ &\leq y_1 \sum_{\beta \in J} (\|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 - \frac{2}{3}) + \frac{2}{3} \sum_{\beta \in J} y_\beta + \sum_{\beta \in S/J} y_\beta \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 \\ &\leq \frac{2}{3} y_1 + \frac{2}{3} \sum_{\beta \in J} y_\beta + \sum_{\beta \in S/J} y_\beta \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 \leq \frac{2}{3} \sum_{\beta=1}^N y_\beta = \frac{2}{3}. \end{aligned}$$

Thus

$$(2.10) \quad a = 0 \quad \text{and} \quad \sum_{\beta \in J} \|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 = \frac{2}{3}(n_1 + 1) \leq n - 2 < \frac{2}{3}N.$$

Hence $S/(J \cup \{1\}) \neq \emptyset$, and the second “ \leq ” in line (2.9) should be “ $<$ ” by the definition of J and the positivity of y_β for $\beta \in S/(J \cup \{1\})$. \square

Now we consider the equality condition of (ii) of Theorem 1.1 in view of the proof of the a priori estimate.

If there is an orthogonal matrix Q and a point $y \in \Delta$ such that $f_Q(y) = 0$, we can assume without loss of generality that

$$y \in \Delta^\gamma := \{x \in \Delta \mid x_\alpha > 0 \text{ for all } \alpha \leq \gamma \text{ and } x_\beta = 0 \text{ for all } \beta > \gamma\}$$

for some $2 \leq \gamma \leq N$. Then y is a maximum point of $f_Q(x)$ in \mathbb{R}_+^N and an interior maximum point in Δ^γ . Therefore, we have the same conclusions as (2.7, 2.8, 2.9, 2.10) when $\gamma \leq n_1 + 1$, and all inequalities in the proof of Lemma 2.3 can be replaced by equalities. So $n_0 = 1$ by Lemma 2.2, $\check{S} = \{(1, 4), (2, 3)\}$, $\check{q}_{ii}^\beta = \check{q}_{ij}^\beta = 0$ for any $(i, j) \in S/\check{S}$, $\beta \in J$, which imply that \tilde{Q}_β is a linear combination of $\text{diag}(D_2, 0)$, $\text{diag}(D_3, 0)$ for any $\beta \in J$. Hence, $1 \leq n_1 \leq 2$. But if $n_1 = 1$, it follows from (2.10) that $\|[\tilde{Q}_1, \tilde{Q}_\beta]\|^2 = \frac{4}{3} > 1$ for $\beta \in J$ which contradicts with Lemma 2.5. So we have $n_1 = 2$ and $2 \leq \gamma \leq 3$. If $\gamma = 2$, then it follows from Lemma 2.5 and (2.9) the following contradiction:

$$\frac{2}{3} = y_2 \|[\tilde{Q}_1, \tilde{Q}_2]\|^2 \leq \frac{1}{2}.$$

So we get $\gamma = 3$. By (2.9) again, we have $y_1 = y_2 = y_3 = \frac{1}{3}$ and $y_\alpha = 0$ for $\alpha > 3$, and

$$\|[\tilde{Q}_1, \tilde{Q}_2]\|^2 = \|[\tilde{Q}_1, \tilde{Q}_3]\|^2 = \|[\tilde{Q}_2, \tilde{Q}_3]\|^2 = 1,$$

from which we can conclude the equality case of (ii) of Theorem 1.1 by Lemmas 2.5 and 2.7. The proof of Theorem 1.1 is now completed. \square

3. SIMONS-TYPE INEQUALITY FOR RIEMANNIAN SUBMERSIONS

3.1. Moving frame method for Riemannian submersions. In this subsection we present a treatment of basic materials about Riemannian submersions by moving frame method.

Let $\pi : M^{n+m} \rightarrow B^n$ be a Riemannian submersion. We denote by D, R, r (resp. $\hat{D}, \hat{R}, \hat{r}; \check{D}, \check{R}, \check{r}$) the Levi-Civita connection, the curvature operator and the Ricci operator on M (resp. on the fibres; on B) respectively. Around each point $x \in M$, we can choose local orthonormal vertical vector fields $\{U_{n+1}, \dots, U_{n+m}\}$ and local orthonormal basic vector fields $\{X_1, \dots, X_n\}$ which are horizontal and projectable such that $\{\check{X}_1 := \pi_* X_1, \dots, \check{X}_n := \pi_* X_n\}$ form a local orthonormal basis around $\pi(x) \in B$. Thus $\{X_1, \dots, X_n, U_{n+1}, \dots, U_{n+m}\}$ form a local orthonormal basis of TM around $x \in M$ and we denote by $\{\omega_1, \dots, \omega_n, \omega_{n+1}, \dots, \omega_{n+m}\}$ the dual 1-forms on M with respect to this basis, *i.e.*,

$$\omega_i(X_j) = \delta_{ij}, \quad \omega_i(U_r) = \omega_r(X_i) = 0, \quad \omega_r(U_s) = \delta_{rs},$$

where, from now on, we use the convention for indices as follows:

$$h, i, j, k, l \in \{1, \dots, n\}; \quad r, s, t, u, v \in \{n+1, \dots, n+m\}, \quad \alpha, \beta, \gamma, \delta \in \{1, \dots, n+m\}.$$

Also we denote by $\{\check{\omega}_1, \dots, \check{\omega}_n\}$ the dual 1-forms on B with respect to the basis $\{\check{X}_1, \dots, \check{X}_n\}$ and by $\{\hat{\omega}_{n+1}, \dots, \hat{\omega}_{n+m}\}$ the dual 1-forms on the fibre(s) with respect to the basis $\{U_{n+1}, \dots, U_{n+m}\}$. Then the connection 1-forms $\{\omega_{\alpha\beta}\}$ of D on M , the connection 1-forms $\{\hat{\omega}_{rs}\}$ of \hat{D} on the fibre(s) and the connection 1-forms $\{\check{\omega}_{ij}\}$ of \check{D} can be defined as follows:

(3.1)

$$\begin{aligned} \omega_{ij} &= \langle DX_i, X_j \rangle, \quad \omega_{ir} = \langle DX_i, U_r \rangle = -\langle DU_r, X_i \rangle = -\omega_{ri}, \quad \omega_{rs} = \langle DU_r, U_s \rangle; \\ \hat{\omega}_{rs} &= \langle \hat{D}U_r, U_s \rangle, \quad \check{\omega}_{ij} = \langle \check{D}\check{X}_i, \check{X}_j \rangle, \end{aligned}$$

where without confusion we denote by bracket simultaneously the metrics on M , B and the fibres. Let $\{\Omega_{\alpha\beta}\}$ (resp. $\{\hat{\Omega}_{rs}\}$; $\{\check{\Omega}_{ij}\}$) be the curvature 2-forms on M (resp. on the fibres; on B). Then we have the following structure equations:

(3.2)

$$\begin{cases} d\omega_\alpha = \omega_{\alpha\beta} \wedge \omega_\beta, & \omega_{\alpha\beta} = -\omega_{\beta\alpha}, \\ d\omega_{\alpha\beta} = \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}; \end{cases}$$

(3.3)

$$\begin{cases} d\hat{\omega}_r = \hat{\omega}_{rs} \wedge \hat{\omega}_s, & \hat{\omega}_{rs} = -\hat{\omega}_{sr}, \\ d\hat{\omega}_{rs} = \hat{\omega}_{rt} \wedge \hat{\omega}_{ts} + \hat{\Omega}_{rs}; \end{cases}$$

(3.4)

$$\begin{cases} d\check{\omega}_i = \check{\omega}_{ij} \wedge \check{\omega}_j, & \check{\omega}_{ij} = -\check{\omega}_{ji}, \\ d\check{\omega}_{ij} = \check{\omega}_{ik} \wedge \check{\omega}_{kj} + \check{\Omega}_{ij}, \end{cases}$$

where, from now on, repeated indices are implicitly summed over, and we will write the curvature forms as $\Omega_{\alpha\beta} = -\frac{1}{2}R_{\alpha\beta\gamma\delta}\omega_\gamma \wedge \omega_\delta$ and so the Ricci curvature $r = (R_{\alpha\beta})$ (resp. $\hat{r} = (\hat{R}_{rs})$; $\check{r} = (\check{R}_{ij})$) on M (resp. on the fibre(s); on B) can be expressed as $R_{\alpha\beta} = R_{\alpha\gamma\beta\gamma}$ (resp. $\hat{R}_{rs} = \hat{R}_{rtst}$; $\check{R}_{ij} = \check{R}_{ikjk}$).

Now since $\pi_*[X_i, U_r] = [\check{X}_i, \pi_*U_r] = 0$ and $\pi_*[U_r, U_s] = [\pi_*U_r, \pi_*U_s] = 0$, $[X_i, U_r]$ and $[U_r, U_s]$ are vertical, thereby it follows from (3.1) and the definitions of the tensors T and A in (1.3) that

(3.5)

$$\begin{aligned} T_{rs}^i &:= \omega_{ri}(U_s) = \langle T_{U_s}U_r, X_i \rangle = -\langle T_{U_s}X_i, U_r \rangle = T_{sr}^i; \\ A_{ij}^r &:= \omega_{ir}(X_j) = \langle A_{X_j}X_i, U_r \rangle = -\langle A_{X_j}U_r, X_i \rangle = \omega_{ij}(U_r) = -A_{ji}^r. \end{aligned}$$

Hence one can see that the tensor T (or its coefficients $\{T_{rs}^i\}$) is just the second fundamental form when it is restricted to vertical vector fields along the fibre(s). Meanwhile, we find that

$$A_{X_i}X_j = -A_{X_j}X_i = \frac{1}{2}\mathcal{V}[X_i, X_j]$$

and thus

$$A_XY = \frac{1}{2}\mathcal{V}[X, Y], \quad \text{for } X, Y \in \mathcal{H},$$

which shows that A measures the integrability of the horizontal distribution \mathcal{H} and so it is usually called the *integrability tensor* of π . By (1.4) and (3.5), we have

(3.6)

$$|A|^2 = \sum_{r,i,j} (A_{ij}^r)^2.$$

Moreover, formulas (3.5) imply the following equations:

$$(3.7) \quad \begin{aligned} \omega_{ir} &= A_{ij}^r \omega_j - T_{rs}^i \omega_s, \\ \omega_{ij} &= \pi^* \tilde{\omega}_{ij} + A_{ij}^r \omega_r. \end{aligned}$$

Define the covariant derivatives of T_{rs}^i and A_{ij}^r by

$$(3.8) \quad \begin{aligned} DT_{rs}^i &:= dT_{rs}^i + T_{ts}^i \omega_{tr} + T_{rt}^i \omega_{ts} + T_{rs}^j \omega_{ji} =: T_{rsj}^i \omega_j + T_{rst}^i \omega_t, \\ DA_{ij}^r &:= dA_{ij}^r + A_{kj}^r \omega_{ki} + A_{ik}^r \omega_{kj} + A_{ij}^s \omega_{sr} =: A_{ijk}^r \omega_k + A_{ijs}^r \omega_s. \end{aligned}$$

Then it is easily seen from (3.5) and (3.8) that

$$(3.9) \quad \begin{aligned} T_{rsj}^i &= \langle (D_{X_j} T)_{U_s} U_r, X_i \rangle = T_{srj}^i, & T_{rst}^i &= \langle (D_{U_t} T)_{U_s} U_r, X_i \rangle = T_{srt}^i, \\ A_{ijk}^r &= \langle (D_{X_k} A)_{X_j} X_i, U_r \rangle = -A_{jik}^r, & A_{ijs}^r &= \langle (D_{U_s} A)_{X_j} X_i, U_r \rangle = -A_{jis}^r, \end{aligned}$$

which are the only components of DT and DA that cannot be recovered from T and A at a point (cf. [2, 19]). Taking deferential of (3.7) by using (3.8) and the structure equations (3.2, 3.4) we get

$$(3.10) \quad (DA_{ij}^r + A_{ik}^r A_{jk}^s \omega_s + T_{rs}^i A_{jk}^s \omega_k) \wedge \omega_j = (DT_{rs}^i - T_{rt}^i T_{ts}^k \omega_k) \wedge \omega_s + \Omega_{ir},$$

$$(3.11) \quad \begin{aligned} \Omega_{ij} &= \pi^* \tilde{\Omega}_{ij} + (A_{ij}^r A_{kl}^r + A_{ik}^r A_{jl}^r) \omega_k \wedge \omega_l \\ &\quad + (A_{ijk}^r - A_{ij}^s T_{sr}^k + A_{jk}^s T_{sr}^i + A_{ki}^s T_{sr}^j) \omega_k \wedge \omega_r \\ &\quad + (A_{ijs}^r + T_{ts}^i T_{tr}^j + A_{ik}^s A_{kj}^r) \omega_s \wedge \omega_r. \end{aligned}$$

Recall that the O'Neill's formula $\{0\}$ in [19] is just the Gauss equation on the fibre(s) derived from the structure equations (3.2, 3.3) and can be written as

$$(3.12) \quad R_{rstu} = \hat{R}_{rstu} - T_{rt}^i T_{su}^i + T_{st}^i T_{ru}^i.$$

Taking values of (3.10) on $U_s \wedge U_t$, $X_j \wedge U_s$ and of (3.11) on $U_s \wedge U_r$, $X_k \wedge U_r$ and $X_k \wedge X_l$, respectively, we can get the O'Neill's formulas $\{1, 2, 2', 3, 4\}$ in [19] as follows:

$$(3.13) \quad R_{irst} = T_{rts}^i - T_{rst}^i,$$

$$(3.14) \quad R_{irjs} = T_{rsj}^i + A_{ijs}^r - T_{rt}^i T_{ts}^j + A_{ik}^r A_{jk}^s,$$

$$(3.15) \quad R_{ijsr} = A_{ijr}^s - A_{ijs}^r + A_{ik}^r A_{kj}^s - A_{ik}^s A_{kj}^r + T_{tr}^i T_{ts}^j - T_{ts}^i T_{tr}^j,$$

$$(3.16) \quad R_{ijk r} = -A_{ijk}^r + A_{ij}^s T_{sr}^k - A_{jk}^s T_{sr}^i - A_{ki}^s T_{sr}^j,$$

$$(3.17) \quad R_{ijkl} = \tilde{R}_{ijkl} \circ \pi - 2A_{ij}^r A_{kl}^r - A_{ik}^r A_{jl}^r + A_{il}^r A_{jk}^r.$$

Taking value of (3.10) on $X_j \wedge X_k$ we get

$$R_{irjk} = A_{ijk}^r - A_{ikj}^r + 2A_{jk}^s T_{rs}^i,$$

which by combining with (3.9, 3.16) implies

$$(3.18) \quad A_{ijk}^r + A_{jki}^r + A_{kij}^r = A_{ji}^s T_{sr}^k + A_{kj}^s T_{sr}^i + A_{ik}^s T_{sr}^j.$$

Reversing i and j , r and s in (3.14) and using (3.9) and the symmetry of the curvature operator, we can get the following (cf. [2, 14]):

$$(3.19) \quad A_{ijs}^r + A_{ijr}^s = T_{rsi}^j - T_{rsj}^i.$$

Let $\{K_{\alpha\beta}\}$ (resp. $\{\hat{K}_{rs}\}$; $\{\check{K}_{ij}\}$) be the sectional curvatures of M (resp. of the fibre(s); of B). Then it follows from (3.12-3.17) that

$$(3.20) \quad \begin{aligned} K_{rs} &= \hat{K}_{rs} + \sum_i \left((T_{rs}^i)^2 - T_{rr}^i T_{ss}^i \right), \\ K_{ir} &= T_{rri}^i - \sum_s (T_{rs}^i)^2 + \sum_j (A_{ij}^r)^2, \\ K_{ij} &= \check{K}_{ij} \circ \pi - 3 \sum_r (A_{ij}^r)^2, \end{aligned}$$

where, unusually, repeated indices are not summed over. If the fibres are totally geodesic, *i.e.*, $T = 0$, then by (3.12-3.17) we have the following identities about Ricci curvatures:

$$(3.21) \quad \begin{aligned} R_{ir} &= A_{ikk}^r = -\langle \check{\delta} A(X_i), U_r \rangle, \\ R_{rs} &= \hat{R}_{rs} + A_{ij}^r A_{ij}^s, \\ R_{ij} &= \check{R}_{ij} \circ \pi - 2A_{ik}^r A_{jk}^r. \end{aligned}$$

Hence if M is Einstein with totally geodesic fibres, then we have

$$(3.22) \quad R_{ir} = A_{ikk}^r = -\langle \check{\delta} A(X_i), U_r \rangle = 0,$$

which is equivalent to that the horizontal distribution \mathcal{H} is Yang-Mills.

3.2. Laplacians of the integrability tensor. From now on, we assume that the Riemannian submersion $\pi : M^{n+m} \rightarrow B^n$ has totally geodesic fibres and Yang-Mills horizontal distribution, *i.e.*, $T = 0$ and $A_{ikk}^r = 0$ (by (3.22)).

We define the covariant derivatives of A_{ijk}^r and A_{ijs}^r by

$$(3.23) \quad \begin{aligned} DA_{ijk}^r &:= dA_{ijk}^r + A_{lj}^r \omega_{li} + A_{il}^r \omega_{lj} + A_{ijl}^r \omega_{lk} + A_{ijk}^s \omega_{sr} =: A_{ijkl}^r \omega_l + A_{ijks}^r \omega_s, \\ DA_{ijs}^r &:= dA_{ijs}^r + A_{kjs}^r \omega_{ki} + A_{iks}^r \omega_{kj} + A_{ijt}^r \omega_{ts} + A_{ijs}^t \omega_{tr} =: A_{ijsk}^r \omega_k + A_{ijst}^r \omega_t. \end{aligned}$$

The horizontal and vertical Laplacians of A_{ij}^r are defined by

$$(3.24) \quad \Delta^{\mathcal{H}} A_{ij}^r := A_{ijkk}^r, \quad \Delta^{\mathcal{V}} A_{ij}^r := A_{ijss}^r,$$

while the horizontal and vertical Laplacians of a function $f \in C^\infty(M)$ are defined by

$$(3.25) \quad \Delta^{\mathcal{H}} f := (X_i X_i - D_{X_i} X_i) f, \quad \Delta^{\mathcal{V}} f := (U_s U_s - D_{U_s} U_s) f.$$

It is easily seen that these Laplacians are well-defined and relate to the Laplace-Beltrami operator Δ of M by

$$\Delta = \Delta^{\mathcal{H}} + \Delta^{\mathcal{V}}.$$

Moreover, since the fibres are totally geodesic, $\Delta^\mathcal{V}$ is just the Laplace-Beltrami operator, also denoted by Δ , along any fibre F_b when restricted to actions on functions of F_b , *i.e.*,

$$(\Delta^\mathcal{V} f)|_{F_b} = \Delta(f|_{F_b}), \quad \text{for any } f \in C^\infty(M).$$

Therefore, if M is closed, then for any function $f \in C^\infty(M)$, we have

$$(3.26) \quad \int_M \Delta^\mathcal{H} f \, dV_M = 0, \quad \int_M \Delta^\mathcal{V} f \, dV_M = 0.$$

Taking differential of the second equation of (3.8) by using (3.8, 3.23) and the structure equations (3.2) we get

$$(3.27) \quad \begin{aligned} & DA_{ijk}^r \wedge \omega_k + DA_{ijs}^r \wedge \omega_s \\ = & -(A_{hj}^r A_{hk}^s A_{il}^s + A_{ih}^r A_{hk}^s A_{jl}^s + A_{hl}^r A_{hk}^s A_{ij}^s + A_{ijs}^r A_{kl}^s) \omega_k \wedge \omega_l \\ & - A_{ijl}^r A_{lk}^s \omega_k \wedge \omega_s + (A_{hj}^r \Omega_{hi} + A_{ih}^r \Omega_{hj} + A_{ij}^s \Omega_{sr}). \end{aligned}$$

Evaluating (3.27) on $X_k \wedge X_l$ and $U_s \wedge U_t$, respectively, we obtain

$$(3.28) \quad \begin{aligned} & A_{ijlk}^r - A_{ijkl}^r \\ = & -(A_{hj}^r A_{hk}^s A_{il}^s + A_{ih}^r A_{hk}^s A_{jl}^s + A_{hl}^r A_{hk}^s A_{ij}^s + 2A_{ijs}^r A_{kl}^s) \\ & + (A_{hj}^r A_{hl}^s A_{ik}^s + A_{ih}^r A_{hl}^s A_{jk}^s + A_{hk}^r A_{hl}^s A_{ij}^s) \\ & - (A_{hj}^r R_{hikl} + A_{ih}^r R_{hjk l} + A_{ij}^s R_{srkl}), \end{aligned}$$

$$(3.29) \quad A_{ijts}^r - A_{ijst}^r = -(A_{hj}^r R_{hist} + A_{ih}^r R_{hjst} + A_{ij}^u R_{urst}).$$

Now since $T = 0$ and $A_{ikk}^r = 0$, by combining the identities (3.5, 3.9, 3.15, 3.17, 3.18, 3.19, 3.24) with (3.28, 3.29), we can calculate the Laplacians of the integrability tensor A as follows:

$$(3.30) \quad \begin{aligned} \langle A, \Delta^\mathcal{H} A \rangle &:= A_{ij}^r (\Delta^\mathcal{H} A_{ij}^r) = A_{ij}^r A_{ijkk}^r \\ &= A_{ij}^r (-A_{jik}^r - A_{kij}^r) = 2A_{ij}^r A_{ikjk}^r = 2A_{ij}^r (A_{ikjk}^r - A_{ikkj}^r) \\ &= 2A_{ij}^r \left(-(A_{hk}^r A_{hk}^s A_{ij}^s + 2A_{iks}^r A_{kj}^s) + 2A_{hk}^r A_{hj}^s A_{ik}^s \right. \\ &\quad \left. - (A_{hk}^r R_{hikj} + A_{ih}^r R_{hkkj} + A_{ik}^s R_{srkj}) \right) \\ &= 2A_{ij}^r \left(2A_{ih}^r A_{hk}^s A_{kj}^s + 2A_{hk}^r A_{hj}^s A_{ik}^s \right. \\ &\quad \left. - (A_{hk}^r \check{R}_{hikj} \circ \pi + A_{ih}^r \check{R}_{hkkj} \circ \pi) - 2A_{ik}^s R_{srkj} \right) \\ &= -2\| [A^r, A^s] \|^2 - A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi + 2A_{ij}^r A_{ih}^r \check{R}_{jih} \circ \pi - 4A_{ij}^r A_{ik}^s R_{srkj}, \end{aligned}$$

$$\begin{aligned}
(3.31) \quad \langle A, \Delta^{\mathcal{V}} A \rangle &:= A_{ij}^r (\Delta^{\mathcal{V}} A_{ij}^r) = A_{ij}^r A_{ijss}^r \\
&= -A_{ij}^r A_{ijrs}^s = A_{ij}^r (A_{ijsr}^s - A_{ijrs}^s) \\
&= -A_{ij}^r (A_{hj}^s R_{hirs} + A_{ih}^s R_{hjrs} + A_{ij}^u R_{usrs}) \\
&= 2A_{ij}^r A_{ik}^s R_{srkj} - A_{ij}^r A_{ij}^s \hat{R}_{rs},
\end{aligned}$$

where we denote by $A^r := (A_{ij}^r)$ the $(n \times n)$ skew-symmetric matrix corresponding to the operator $AU_r : TM \rightarrow TM$ defined by $AU_r(X_i) := A_{X_i}U_r = A_{ij}^r X_j$, and the square norm of the Lie bracket in the last line of (3.30) is implicitly summed over all the indices r and s .

3.3. Simons-type inequality. In this subsection we will derive the Simons-type inequality rendered in Theorem 1.2 for Riemannian submersions with totally geodesic fibres and Yang-Mills horizontal distributions.

We denote by $\nabla^{\mathcal{H}}$ (resp. $\nabla^{\mathcal{V}}$) the restriction to the horizontal (resp. vertical) distribution of the covariant derivative D on M , i.e.,

$$\nabla^{\mathcal{H}} W := (DW)|_{\mathcal{H}}, \quad \nabla^{\mathcal{V}} W := (DW)|_{\mathcal{V}}, \quad \text{for any tensor } W \text{ on } M.$$

From (3.6, 3.24, 3.25) we can derive the following

$$(3.32) \quad \frac{1}{2} \Delta^{\mathcal{H}} |A|^2 = \langle A, \Delta^{\mathcal{H}} A \rangle + |\nabla^{\mathcal{H}} A|^2, \quad \frac{1}{2} \Delta^{\mathcal{V}} |A|^2 = \langle A, \Delta^{\mathcal{V}} A \rangle + |\nabla^{\mathcal{V}} A|^2.$$

Combining (3.15, 3.16, 3.19, 3.28, 3.29, 3.32) we obtain

$$\begin{aligned}
(3.33) \quad & \left(\frac{1}{2} \Delta^{\mathcal{H}} + 2 \Delta^{\mathcal{V}} \right) |A|^2 \\
&= -2 \|[A^r, A^s]\|^2 - A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi + 2A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi - 4A_{ij}^r A_{ij}^s \hat{R}_{rs} \\
&\quad + 4A_{ij}^r A_{ik}^s R_{srkj} + |A_{ijk}^r|^2 + 4|A_{ijs}^r|^2 \\
&= -\|[A^r, A^s]\|^2 - A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi + 2A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi - 4A_{ij}^r A_{ij}^s \hat{R}_{rs} \\
&\quad + |R_{ijk}^r|^2 + |R_{srij}|^2,
\end{aligned}$$

where, from now on, the indices within square norms are also implicitly summed over. If M is closed, then by (3.26, 3.33) we get

$$(3.34) \quad \int_M \left(\|[A^r, A^s]\|^2 + 4A_{ij}^r A_{ij}^s \hat{R}_{rs} + A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi - 2A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi \right) dV_M \geq 0.$$

As defined before Theorem 1.2 in Section 1, for $x \in M$, $\check{\kappa}(x)$ is the largest eigenvalue of the curvature operator \check{R} of B at $\pi(x) \in B$, $\check{\lambda}(x)$ is the lowest eigenvalue of the Ricci curvature \check{r} of B at $\pi(x) \in B$ and $\hat{\mu}(x)$ is the largest eigenvalue of the Ricci curvature \hat{r} of the fibre at x . Then the inequality (3.34) induces the following:

$$(3.35) \quad \int_M \left(\|[A^r, A^s]\|^2 + 4\hat{\mu}|A|^2 + 2\check{\kappa}|A|^2 - 2\check{\lambda}|A|^2 \right) dV_M \geq 0.$$

When $n = 2$, it is obvious that $[A^r, A^s] = 0$ and $\check{\kappa} = \check{\lambda}$. Thus by (3.35) we have

$$\int_M |A|^2 \hat{\mu} \, dV_M \geq 0,$$

which verifies the first case (i) of Theorem 1.2.

When $m = 1$, the first two terms of (3.35) vanish and thus

$$\int_M |A|^2 (\check{\kappa} - \check{\lambda}) \, dV_M \geq 0,$$

which proves the second case (ii) of Theorem 1.2.

Now we are coming to discover the phenomenons of “duality” between symmetric matrices and skew-symmetric matrices, between submanifold geometry and Riemannian submersions, as well as their interactions. To do this, one needs only to apply the inequalities of Theorem 1.1 to (3.35) with the skew-symmetric matrices $\{A^r\}$ instead of $\{B_r\}$, keeping in mind how the algebraic DDVV inequality (of symmetric matrices) applies to prove the Simons inequality in submanifold geometry (cf. [17]). This completes the proof of the left two cases (iii, iv) of Theorem 1.2 immediately.

3.4. Equality conclusions. In this subsection we will complete the proof of Theorem 1.2 by verifying the conclusions (a-d) for equality conditions of the Simons-type inequality case by case.

Firstly, it is a well-known fact that the total space M of a Riemannian submersion with vanishing T and A is (at least locally) a Riemannian product $B \times F$, and vice versa. Henceforth, we assume that $A \neq 0$. The proof of (a-d) of Theorem 1.2 goes on as follows:

(a) In each case of (i-iv) of Theorem 1.2, the equality assumption of the integral inequality compels (3.34) to attain its equality simultaneously, which then by (3.26, 3.33) shows immediately

$$(3.36) \quad R_{ijk} \equiv 0, \quad R_{srij} \equiv 0.$$

Now since the fibres are totally geodesic, the Ricci equation on any fibre F_b shows that the normal curvature \hat{R}_{srij}^\perp of F_b equals R_{srij} and thus vanishes. So each fibre has flat normal bundle in M . Moreover, it follows from (3.15, 3.16, 3.19, 3.36) that

$$(3.37) \quad A_{ijk}^r = 0, \quad A_{ijs}^r = \frac{1}{2}[A^r, A^s]_{ij}.$$

Noticing that the covariant derivative of $|A|^2$ can be calculated from (3.37) as

$$D|A|^2 = 2A_{ij}^r A_{ijk}^r \omega_k + 2A_{ij}^r A_{ijs}^r \omega_s = 0,$$

we arrive at the conclusion that $|A|^2 \equiv \text{Const} =: C > 0$. Then by (3.30-3.32) and (3.36, 3.37), we have

$$(3.38) \quad \frac{1}{2} \Delta^{\mathcal{H}} |A|^2 = -2\| [A^r, A^s] \|^2 - A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi + 2A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi \equiv 0,$$

$$(3.39) \quad \frac{1}{2} \Delta^{\mathcal{V}} |A|^2 = -A_{ij}^r A_{ij}^s \hat{R}_{rs} + \frac{1}{4} \| [A^r, A^s] \|^2 \equiv 0.$$

Now we come to prove the subcases (a1-a4) of (a) as follows.

(a1) Now $n = 2$ and $[A^r, A^s] \equiv 0$. So by the definition of $\hat{\mu}$ and (3.39), we get

$$|A|^2 \hat{\mu} \geq A_{ij}^r A_{ij}^s \hat{R}_{rs} = 0,$$

whereas $|A|^2 \equiv C > 0$ and $\int_M |A|^2 \hat{\mu} dV_M = 0$ by assumption.

This proves that $\hat{\mu} \equiv 0$.

(a2) Now $m = 1$ and $[A^r, A^s] \equiv 0$. So by the definitions of $\check{\kappa}$, $\check{\lambda}$ and (3.38), we get

$$|A|^2 (\check{\kappa} - \check{\lambda}) \geq \frac{1}{2} A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi - A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi = 0,$$

whereas $|A|^2 \equiv C > 0$ and $\int_M |A|^2 (\check{\kappa} - \check{\lambda}) dV_M = 0$ by assumption.

This proves that $\check{\kappa} - \check{\lambda} \equiv 0$.

(a3) Now the equality assumption implies that the inequality in (i) of Theorem 1.1 (with $B_r = A^r$) attains its equality, *i.e.*,

$$(3.40) \quad \sum_{r,s} \| [A^r, A^s] \|^2 = \frac{1}{3} \left(\sum_r |A^r|^2 \right)^2 = \frac{1}{3} |A|^4 = \frac{1}{3} C^2.$$

Then by the definitions of $\hat{\mu}$, $\check{\kappa}$, $\check{\lambda}$ and (3.38, 3.39), we have

$$|A|^2 \hat{\mu} \geq A_{ij}^r A_{ij}^s \hat{R}_{rs} = \frac{1}{4} \| [A^r, A^s] \|^2 = \frac{1}{12} C^2,$$

$$|A|^2 (\check{\kappa} - \check{\lambda}) \geq \frac{1}{2} A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi - A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi = -\| [A^r, A^s] \|^2 = -\frac{1}{3} C^2,$$

whereas $|A|^2 \equiv C > 0$ and $\int_M |A|^2 \left(\frac{1}{6} |A|^2 + 2\hat{\mu} + \check{\kappa} - \check{\lambda} \right) dV_M = 0$ by assumption.

This proves that $\hat{\mu} \equiv \frac{1}{12} C$, $\check{\kappa} - \check{\lambda} \equiv -\frac{1}{3} C$.

(a4) The proof is almost the same with that of (a3) except for that the coefficient $\frac{1}{3}$ in (3.40) would be substituted by $\frac{2}{3}$. So we omit it here.

(b) If the equality in (iii) (resp. (iv)) holds, as in the proof of (a3), the inequality in (i) (resp. (ii)) of Theorem 1.1 (with $B_r = A^r$) attains its equality, thereby, under some $K = O(n) \times O(m)$ action which can be realized by a choice of an orthonormal horizontal basis $\{X_1, \dots, X_n\}$ and of an orthonormal vertical basis $\{U_{n+1}, \dots, U_{n+m}\}$, the matrices A^r 's are all equal to zero except $A^{n+1}, A^{n+2}, A^{n+3}$, which are in the forms of C_1, C_2, C_3 (resp. $\text{diag}(D_1, 0), \text{diag}(D_2, 0), \text{diag}(D_3, 0)$). Noticing that now we have

$$|A|^2 = |A^{n+1}|^2 + |A^{n+2}|^2 + |A^{n+3}|^2 \equiv C > 0,$$

we derive that $m \geq 3$. Moreover, we can rewrite $A^{n+1}, A^{n+2}, A^{n+3}$ as follows:

$$(3.41) \quad \begin{aligned} A^{n+1} &= \sqrt{\frac{C}{6}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{n+2} = \sqrt{\frac{C}{6}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ A^{n+3} &= \sqrt{\frac{C}{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{for equality case of (iii);} \end{aligned}$$

$$(3.42) \quad \begin{aligned} A^{n+1} &= \sqrt{\frac{C}{12}} \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array} \right), \quad A^{n+2} = \sqrt{\frac{C}{12}} \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right), \\ A^{n+3} &= \sqrt{\frac{C}{12}} \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array} \right) \quad \text{for equality case of (iv),} \end{aligned}$$

where 0 in the diagonals of (3.42) is a zero matrix of order $(n-4)$. As in the proof of (a3), we have the following equations if the equality in (iii) or (iv) holds:

$$(3.43) \quad |A|^2 \hat{\mu} = A_{ij}^r A_{ij}^s \hat{R}_{rs}, \quad |A|^2 \check{\kappa} = \frac{1}{2} A_{ij}^r A_{hk}^r \check{R}_{ijhk} \circ \pi, \quad |A|^2 \check{\lambda} = A_{ij}^r A_{ih}^r \check{R}_{jh} \circ \pi.$$

Using the formulas (3.41) for equality case of (iii), the equations (3.43) can be turned to the following:

$$\begin{aligned} \hat{\mu} &= \frac{1}{3}(\hat{R}_{n+1 \ n+1} + \hat{R}_{n+2 \ n+2} + \hat{R}_{n+3 \ n+3}), \\ \check{\kappa} &= \frac{1}{3}(\check{R}_{1212} \circ \pi + \check{R}_{1313} \circ \pi + \check{R}_{2323} \circ \pi), \\ \check{\lambda} &= \frac{1}{3}(\check{R}_{11} \circ \pi + \check{R}_{22} \circ \pi + \check{R}_{33} \circ \pi). \end{aligned}$$

Then recalling the definitions of $\hat{\mu}, \check{\kappa}, \check{\lambda}$, we obtain the following decompositions for $\hat{r}, \check{R}, \check{r}$ for equality case of (iii):

$$\hat{r} = \hat{\mu} I_3 \oplus \hat{r}', \quad \check{R} \equiv \check{\kappa} I_3, \quad \check{r} \equiv \check{\lambda} I_3,$$

where $\hat{r}' = \hat{r}|_{\text{span}\{U_7, \dots, U_{3+m}\}}$ if $m \geq 4$ and 0 if $m = 3$, $\check{\lambda} = 2\check{\kappa}$ because of $n = 3$ now.

Similarly, using the formulas (3.42) for equality case of (iv) and the first Bianchi identity, the equations (3.43) can be turned to the following:

$$\begin{aligned} \hat{\mu} &= \frac{1}{3}(\hat{R}_{n+1 \ n+1} + \hat{R}_{n+2 \ n+2} + \hat{R}_{n+3 \ n+3}), \\ \check{\kappa} &= \frac{1}{6}(\check{R}_{1212} \circ \pi + \check{R}_{1313} \circ \pi + \check{R}_{1414} \circ \pi + \check{R}_{2323} \circ \pi + \check{R}_{2424} \circ \pi + \check{R}_{3434} \circ \pi), \\ \check{\lambda} &= \frac{1}{4}(\check{R}_{11} \circ \pi + \check{R}_{22} \circ \pi + \check{R}_{33} \circ \pi + \check{R}_{44} \circ \pi). \end{aligned}$$

Then recalling the definitions of $\hat{\mu}, \check{\kappa}, \check{\lambda}$, we obtain the following decompositions for $\hat{r}, \check{R}, \check{r}$ for equality case of (iv):

$$\hat{r} = \hat{\mu} I_3 \oplus \hat{r}', \quad \check{R} = \check{\kappa} I_6 \oplus \check{R}', \quad \check{r} \equiv \check{\lambda} I_4 \oplus \check{r}',$$

where $\hat{r}' = \hat{r}|_{\text{span}\{U_{n+4}, \dots, U_{n+m}\}}$ if $m \geq 4$ and 0 if $m = 3$, $\check{R}' = \check{R}|_{\text{span}\{X_i \wedge X_j | 1 \leq i \leq n, 5 \leq j \leq n\}}$ and $\check{r}' = \check{r}|_{\text{span}\{X_5, \dots, X_n\}}$ if $n \geq 5$ and 0 if $n = 4$.

From the decompositions, if $m = 3$, then we can see that the 3-dimensional fibres have constant Ricci curvature and thus have constant sectional curvature; if $n = 3$ or 4, then the base manifold B^n has constant sectional curvature; if $n = 5$, then by the definitions of $\check{\kappa}, \check{\lambda}$ we have

$$\begin{aligned} \check{\lambda} &\leq \check{R}_{55} = \check{R}_{1515} + \check{R}_{2525} + \check{R}_{3535} + \check{R}_{4545} \leq 3\check{\kappa} + \check{R}_{i5i5}, \\ \check{\lambda} &= \check{R}_{ii} = \sum_{j=1}^5 \check{R}_{ijij} = 3\check{\kappa} + \check{R}_{i5i5}, \quad \text{for } i = 1, 2, 3, 4. \end{aligned}$$

These prove that $\check{R}_{i5i5} = \check{\kappa}$ for $i = 1, 2, 3, 4$, and so the base manifold B^5 has constant sectional curvature.

(c) Now $m = 3, n = 3$ and the equality in (iii) holds. In (b) we have proved that both of the fibres and the base manifold B^3 have constant sectional curvature. Due to a result of Hermann [15] we see that the fibres are all isometric. Reset $|A|^2 \equiv C =: 24a > 0$, then by (a3) and (b) we get

$$\hat{\mu} = 2a, \quad \check{\lambda} = 2\check{\kappa} = 16a,$$

which deduce the conclusions of (c1) and (c2).

The identities in (c3) can be calculated from the formulas (3.20, 3.21, 3.41). In fact, since we have $T = 0$ and $A_{ikk} = 0$, the formulas (3.20, 3.21) turn into the following:

$$(3.44) \quad \begin{aligned} K_{rs} &= \hat{K}_{rs}, \quad K_{ir} = \sum_j (A_{ij}^r)^2, \quad K_{ij} = \check{K}_{ij} \circ \pi - 3 \sum_r (A_{ij}^r)^2; \\ R_{ir} &= 0, \quad R_{rs} = \hat{R}_{rs} + A_{ij}^r A_{ij}^s, \quad R_{ij} = \check{R}_{ij} \circ \pi - 2A_{ik}^r A_{jk}^r. \end{aligned}$$

Then using formulas (3.41, 3.44) and the known facts that $\hat{K}_{rs} = a$, $\check{K}_{ij} = 8a$, we complete the proof. One should notice that the index range for r in (c3) is $\{1, 2, 3\}$ rather than $\{n+1, n+2, n+3\}$ ($n = 3$) here.

(d) Based on results of (b) and formulas (3.42, 3.44), the proof of the assertions for (d2) and the heading paragraph of (d) are exactly the same with that of (c) despite that we reset $|A|^2 \equiv C =: 12a > 0$ here in view of (a4). As for (d1), we first calculate the sectional curvatures of B^4 and M^7 respectively and find that B has constant sectional curvature $4a$ and M has constant sectional curvature a . In fact, by (a4), (b) and (3.42, 3.44) we know that

$$\hat{\mu} = 2a, \quad \check{\lambda} = 3\check{\kappa} = 12a, \quad K_{rs} = K_{ir} = K_{ij} = a.$$

Hence, M^7 is covered by $S^7(\frac{1}{\sqrt{a}})$, B^4 is covered by $S^4(\frac{1}{2\sqrt{a}})$ and we denote by π_1, π_2 the corresponding covering maps. Thus there is a Riemannian submersion $\pi_0 : S^7(\frac{1}{\sqrt{a}}) \rightarrow S^4(\frac{1}{2\sqrt{a}})$ (lift map of $\pi \circ \pi_1$ through π_2) such that $\pi_2 \circ \pi_0 = \pi \circ \pi_1$. Recall that Ranjan [22] showed that $\pi_0 : S^7(\frac{1}{\sqrt{a}}) \rightarrow S^4(\frac{1}{2\sqrt{a}})$ is equivalent to the Hopf fibration

(see also [11]). Without loss of generality, we can assume that π_0 is just the Hopf fibration, since otherwise we can alter π_1, π_2 by taking compositions with corresponding isometries (bundle isometry between π_0 and the Hopf fibration) of $S^7(\frac{1}{\sqrt{a}})$ and $S^4(\frac{1}{2\sqrt{a}})$ respectively. The proof of (d1) is now completed.

In conclusion, the proof of Theorem 1.2 is now completed.

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